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Tight Framelets with Differential Relations on the Interval*

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Abstract: By employing the refinable matrix of the scaling function, the method for constructing a pair of tight framelets with differential relations on the interval is given in this paper. In which the tight framelets is defined in the sense of Chui's definition. Moreover, an explicit example is constructed by using B-spline. This result can be applied to solve some differential equations.

Keywords: tight framelets; B-spline; differential relations

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1 Introduction

Divergence free wavelets have some applications for the analysis and numerical simulation of incompressible flows typically modeled by the Stokes system or by the incompressible Navier-Stokes equations[1]. In order to find the kind of wavelets over bounded domain, two families of wavelets with differential relations on the interval are needed firstly. Although there exist many wavelets on the interval[2-4], none of them has differential relations. Lakey and Pereyra[5] constructed wavelets on the interval with differential relations by using the multiwavelets of $L_2(\mathbf{R})$, which was introduced by Hardin and Marasovich[6]. More precisely, by truncating the scaling functions and the wavelets of $L_2(\mathbf{R})$, they obtained a pair of biorthogonal MRA's and wavelets for $L_2[0,1]$. Then they found another pair of MRA's and wavelets by smoothing and roughing method. It turns out that the two pairs of wavelets and scaling functions have differential relations. However, the corresponding divergence free multiwavelets have too many generators and duals, which cause problems in applications.

Framelet is a widely accepted tool in signal and image processing as well as in numerical simulation. In particular, they perform better effects than wavelets in many fields of image processing. Ron and Shen[7] introduced framelets MRA of $L_2(\mathbf{R}^d)$ with the setting of shift-invariant subspaces, and give the filter bank algorithms of the pyramidal decomposition and reconstruction. Since the construction of tight framelets over bounded domain is quite different from that over unbounded domain, Chui *et al*[8] developed a general theory for construction of tight framelets over a bounded interval using univariate splines. Due to the theory, this paper constructs framelets with differential relations on the interval.

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2 Tight Framelets on the interval

In this section, we introduce some notations and preliminary results, which will be used later on. To construct tight frames, we need a sequence of nested subspaces $\{V_j\}_{j\in\mathbf{N}}$ in $L_2(I)$ which satisfies

$$V_1 \subset V_2 \subset \cdots \subset V_j \subset \cdots \to L_2(I), \quad \text{and} \quad \overline{\bigcup_{j\geqslant 1} V_j} = L_2(I),$$

where V_j is linearly spanned by $\phi_{j,1},\cdots,\phi_{j,m_j}$. Although this sequence does not have both translation and dilation invariant properties, it generates a multiresolution analysis (MRA) over bounded domain I. Let Φ_j denote the column vector $[\phi_{j,1},\cdots,\phi_{j,m_j}]^T$ and $\mathbf{M}_j=\{1,\cdots,m_j\}$. Then there exists a matrix P_j of size $m_j\times m_{j+1}$ such that $\Phi_j = P_j\Phi_{j+1}$ due to $V_j\subset V_{j+1}$. The matrix P_j is called a refinable matrix. A function family $\{\Phi_j\}_{j\geqslant 1}$ is said to be locally supported, if the sequence

$$h(\Phi_j) := \max_{k\in m_j} \text{length}(\text{supp}\phi_{j,k})$$

converges to zero.

The next two definitions[8] are also needed in our paper.

Definition 2.1 Let Φ_j be a finite family with cardinality m_j in $L_2(I)$. For any symmetric positive semi-definite (spsd) matrix $S_j=[s_{k,\ell}^{(j)}]_{k,\ell\in m_j}$, the quadratic form T_j is defined by

$$T_j f := \left[\langle f,\phi_{j,k}\rangle\right]_{k\in m_j}^T S_j \left[\langle f,\phi_{j,k}\rangle\right]_{k\in m_j}, \quad f\in L_2(I), \tag{1}$$

and the corresponding kernel

$$K_{S_j}(x,y) := \sum_{k,\ell\in m_j} s_{k,\ell}^{(j)}\phi_{j,k}(x)\phi_{j,\ell}(y).$$

Definition 2.2 Assume that $\{\Phi_j\}_{j\geqslant 1}$ is a locally supported family and S_1 is an spsd matrix, which defines the quadratic form T_1 in (1). Then the family $\{\Psi_j\}_{j\geqslant 1}=\{Q_j\Phi_{j+1}\}_{j\geqslant 1}$ constitutes an MRA tight framelets of $L_2(I)$ with respect to S_1, if

$$T_1 f + \sum_{j\geqslant 1}\sum_{k\in\mathbf{N}_j} |\langle f,\psi_{j,k}\rangle|^2 = \|f\|^2, \quad \forall\, f\in L_2(I). \tag{2}$$

We find that the ground level S_1 is relevant to the order of vanishing moments of the framelets $\Psi_{j,k}$. The following theorem tells us when $\Psi_{j,k}$ constitutes tight framelets.

Theorem 2.1[8] Let $\{\Phi_j\}_{j\geqslant 1}$ be a locally supported family and S_1 an spsd matrix such that $\|T_1 f\| \leqslant \|f\|^2$ for all $f\in L_2(I)$. Then $\{\Psi_j\}_{j\geqslant 1}=\{Q_j\Phi_{j+1}\}$ forms an MRA tight framelets with respect to S_1, if and only if there exist spsd matrices S_j of dimensions $m_j\times m_j$, $j\geqslant 1$, such that the following conditions hold:

(i) The quadratic forms T_j in (1) satisfy

$$\lim_{j\to\infty} T_j f = \|f\|^2, \quad f\in L_2(I); \tag{3}$$

(ii) For each $j\geqslant 1$, Q_j, S_j and S_{j+1} satisfy the identity

$$S_{j+1} - P_j^T S_j P_j = Q_j^T Q_j. \tag{4}$$

Remark 2.1 The condition

$$\lim_{j\to\infty}\int_{|x-y|>\varepsilon}|K_{S_j}(x,y)|\mathrm{d}y=0 \tag{5}$$

for any $\varepsilon > 0$ is sufficient for property (i) in Theorem 2.1. If the matrices S_j have a fixed maximal bandwidth $r > 0$ and $\{\Phi_j\}_{j\geqslant 1}$ is locally supported, then (5) holds, since the integral in (5) is zero for sufficient large j.

Remark 2.2 If $S_j = I_j$ for $j \in \mathbf{N}$, then (4) reduces to $I_{m_{j+1}} - P_j^T P_j = Q_j^T Q_j$. In this situation, Lai and Nam[9] gave a simple construction of tight framelets on the interval. However, their framelets don't have any vanishing moments.

3 Differential relation

This part is devoted to construct two families of tight framelets with differential relations on the interval. Let $\{V_j^+\}_{j\geqslant 1}$ and $\{V_j^-\}_{j\geqslant 1}$ be two nested subspaces of $L_2(I)$, the corresponding generators are Φ_j^+ and Φ_j^-, respectively. We are interested in the differential relation $\Phi_j^{+\prime} = M_j\Phi_j^-$, where M_j is an $m_j^+ \times m_j^-$ matrix, $\Phi_j^+ = [\phi_{j,1},\cdots,\phi_{j,m_j}]^T$ and $\Phi_j^{+\prime} = [\phi_{j,1}^{+\prime},\phi_{j,2}^{+\prime},\cdots,\phi_{j,m_j}^{+\prime}]^T$. Since $\Phi_j^+ = P_j^+\Phi_{j+1}^+$, that

$$\Phi_j^{+\prime} = P_j^+\Phi_{j+1}^{+\prime} = P_j^+ M_{j+1}\Phi_{j+1}^-.$$

On the other hand, $\Phi_j^- = P_j^-\Phi_{j+1}^-$ implies $\Phi_j^{+\prime} = M_j\Phi_j^- = M_jP_j^-\Phi_{j+1}^-$. Suppose $\{\phi_{j+1,k}^-\}_{k\in\mathbf{M}_{j+1}}$ are linearly independent, then we get

$$M_jP_j^- = P_j^+M_{j+1}. \tag{6}$$

Under this assumption, the Gramian matrix $\Gamma_j^- = [\langle\phi_{j,k}^-,\phi_{j,\ell}^-\rangle]_{k,\ell\in m_j}$ is symmetric positive definite, and its dual basis $\widetilde{\Phi}_j^-$ is given by the function vector $\widetilde{\Phi}_j^- = [\widetilde{\phi}_{j,k}^-]_{k\in m_j} = {\Gamma_j^-}^{-1}\Phi_j^-$. It is well known that

$$\|f\|^2 = \left[\langle f,\phi_{j,k}^-\rangle\right]_k^T{\Gamma_j^-}^{-1}\left[\langle f,\phi_{j,k}^-\rangle\right]_k.$$

Now, we give a simple method to construct two families of tight framelets with differential relations on the interval.

Theorem 3.1 Let $\{\Phi_j^+\}_{j\geqslant 1}$ and $\{\Phi_j^-\}_{j\geqslant 1}$ be two locally supported families with differential relation $\Phi_j^{+\prime} = M_j\Phi_j^-$, $\{\Psi_j^+ = Q_j^+\Phi_{j+1}^+\}_{j\geqslant 1}$ be an MRA tight framelets with respect to an spsd matrix S_1^+. Define

$$\Psi_j^- = Q_j^-\Phi_{j+1}^- = \frac{1}{b}Q_j^+M_{j+1}\Phi_{j+1}^-$$

for $j \geqslant 1$, where b is a positive number in $\mathbf{R}$ such that

$${\Gamma_1^-}^{-1} - \frac{1}{b^2}M_1^TS_1^+M_1 \tag{7}$$

is an spsd matrix. If the matrices $S_j^- := \frac{1}{b^2}M_j^TS_j^+M_j$ have fixed maximal bandwidth $r > 0$, then $\{\Psi_j^-\}_{j\geqslant 1}$ constitutes an MRA tight framelets of $L_2(I)$ with respect to S_1^- and $\Psi_j^{+\prime} = b\Psi_j^-$.

Proof Since S_j^+ is an spsd matrix, S_j^- has the same property. Note that

$$\left[\langle f,\phi_{1,k}^-\rangle\right]_{k\in\mathbf{M}_1}^T{\Gamma_1^-}^{-1}\left[\langle f,\phi_{1,k}^-\rangle\right]_{k\in\mathbf{M}_1}$$

is the norm of the orthogonal projection of f onto V_1^-. Then for $f \in L_2(I)$, the condition (7) ensures that

$$T_1^- f = \left[\langle f, \phi_{1,k}^- \rangle\right]_{k\in\mathbf{M}_1}^T S_1^- \left[\langle f, \phi_{1,k}^- \rangle\right]_{k\in\mathbf{M}_1} \leqslant \left[\langle f, \phi_{1,k}^- \rangle\right]_{k\in\mathbf{M}_1}^T {\Gamma_1^-}^{-1} \left[\langle f, \phi_{1,k}^- \rangle\right]_{k\in\mathbf{M}_1} \leqslant \|f\|^2,$$

and T_j^- satisfies (i) of Theorem 2.1 due to the Remark 2.1. Now, we only need to prove S_j^- satisfies (ii) of Theorem 2.1. In fact,

$$\begin{aligned} S_{j+1}^- - {P_j^-}^T S_j^- P_j^- &= \frac{1}{b^2} M_{j+1}^T S_{j+1}^+ M_{j+1} - \frac{1}{b^2} {P_j^-}^T M_j^T S_j^+ M_j P_j^- \\ &= \frac{1}{b^2}\left(M_{j+1}^T S_{j+1}^+ M_{j+1} - M_{j+1}^T {P_j^+}^T S_j^+ P_j^+ M_{j+1}\right) \\ &= \frac{1}{b^2} M_{j+1}^T \left(S_{j+1}^+ - {P_j^+}^T S_j^+ P_j^+\right) M_{j+1} \\ &= \frac{1}{b^2} M_{j+1}^T {Q_j^+}^T Q_j^+ M_{j+1} = {Q_j^-}^T Q_j^-, \end{aligned}$$

where the second equality holds by using the fact (6) and the forth equality due to (ii) of Theorem 2.1. Thus $\{\Phi_j^-\}_{j\geqslant 1}$ constitutes an MRA tight framelets of $L_2(I)$ with respect to S_1^-. Moreover, we have

$$\Psi_j^{+'} = Q_j^+ {\Phi_{j+1}^+}' = Q_j^+ M_{j+1} \Phi_{j+1}^- = b Q_j^- \Phi_{j+1}^- = b\Psi_j^-.$$

This completes the proof.

For the efficiency and simplicity of computation, B-spline is usually used for constructing wavelet functions. Next, we apply B-spline to Theorem 3.1 in order to find the desired tight framelets on the interval.

Let us recall the scaling relation of B-spline ϕ^m for $m \geqslant 2$ (see [10])

$$\phi^m(x) = \sum_{k\in\mathbf{Z}} c_k^m \phi^m(2x-k),$$

where

$$c_k^m = \begin{cases} 2^{-m+1}\binom{m}{k}, & \text{for } 0 \leqslant k \leqslant m; \\ 0, & \text{otherwise.} \end{cases}$$

Consider B-spline function ϕ^m of order m whose dyadic translations are restricted into interval $[0,m]$, i.e., $\phi^m(2^j \cdot -k)|_{[0,m]}$. Assume

$$\phi_{j,k}^m(\cdot) = 2^{j-1}\phi^m(2^{j-1}\cdot -k)|_{[0,m]}, \quad \text{and} \quad V_j^m := \{\phi_{j,k}^m : 1 \leqslant k \leqslant m_j\},$$

then the family of nested sequence $\{V_j^m : j \in \mathbf{Z}_+\}$ forms an MRA of $L_2([0,m])$. Let

$$\Phi_j^m := \left[\phi_{j,1}^m, \cdots, \phi_{j,m_j}^m\right]^T.$$

Then their exists a $m_j \times m_{j+1}$ matrix P_j^m such that $\Phi_j^m = P_j^m \Phi_{j+1}^m$ for each $j \in \mathbf{N}$. In [9], Lai and Nam gave a method to construct tight framelets in this situation. We show the quadratic B-spline and its tight framelets of the ground level in Figure 1.

Since $(\phi^m)'(x) = \phi^{m-1}(x) - \phi^{m-1}(x-1)$, $(\Phi_j^m)' := M_j\Phi_j^{m-1}$ with

$$M_j := 2^{j-1}\begin{pmatrix} -1 & & & & \\ 1 & -1 & & & \\ & 1 & -1 & & \\ & & & \ddots & \\ & & & & -1 \\ & & & & 1 \end{pmatrix}_{m_j\times(m-1)_j}.$$

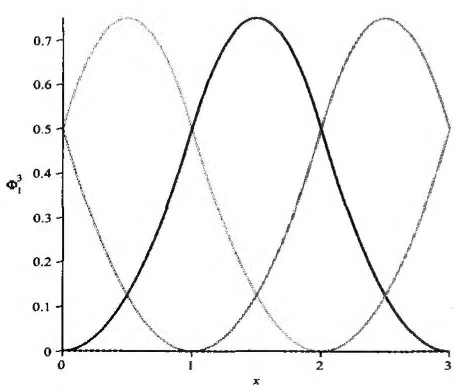
Define $S_j^{m-} := \frac{1}{b^2}M_{j+1}^T M_{j+1}$, where $b \in \mathbf{R}^+$ is chosen such that $\Gamma_1^{m-1} - S_1^{m-}$ is an spsd matrix. Obviously, S_j^{m-} has a fixed maximal bandwidth 2 due to the form of M_j. Hence, we can find the tight framelets

$$\Psi_j^{m-} := \frac{1}{b}Q_j^+ M_{j+1}\Phi_{j+1}^{m-1}.$$

By Theorem 3.1, $\Psi_j^{m\prime} := b\Psi_j^{m-}$. Take $m=3$ and $b=2$, then

$$\Gamma_1^{2-1} - S_1^{2-} = \begin{pmatrix} \frac{37}{15} & -\frac{13}{30} & \frac{4}{15} & -\frac{2}{15} \\ -\frac{13}{30} & \frac{13}{15} & -\frac{1}{30} & \frac{4}{15} \\ \frac{4}{15} & -\frac{1}{30} & \frac{13}{15} & -\frac{13}{30} \\ -\frac{2}{15} & \frac{4}{15} & -\frac{13}{30} & \frac{37}{15} \end{pmatrix}$$

is an spsd matrix. Hence, Ψ_j^3 and Ψ_j^{3-} are tight framelets with differential relation $\Psi_j^{3\prime} = 2\Psi_j^{3-}$. We show linear B-spline and their tight framelets of the ground level in Figure 2.



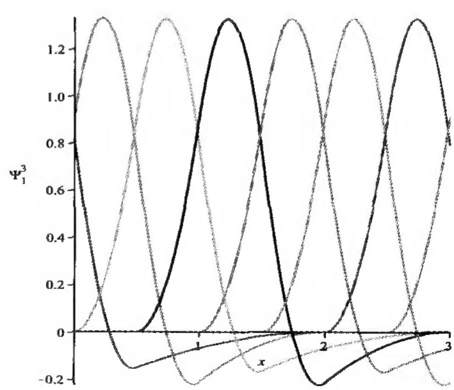
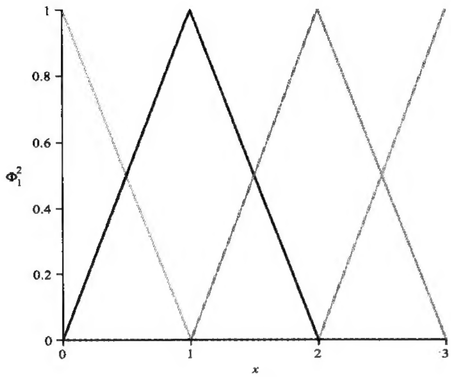


Figure 1: B-spline Φ_1^3 (left) and tight framelets Ψ_1^3 (right)



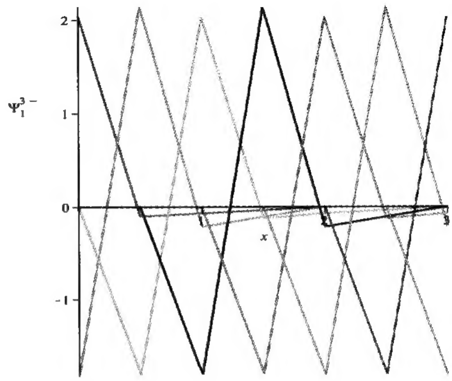


Figure 2: B-spline Φ_1^2 (left) and tight framelets Ψ_1^{3-} (right)

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区间上具有微分关系的紧标架小波

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摘 要: 利用尺度函数的细分矩阵，本文给出了一种构造具有微分关系紧标架小波的方法．其中的紧标架小波是Chui意义下的小波．而且以B-样条为基础，作者构造了具体的例子．此结果可应用于某些微分方程的数值解．

关键词: 紧标架小波；B-样条；微分关系